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Cosmic strings in a product Abelian gauge field theory

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Abstract

It is shown that multiply distributed cosmic strings arise in the product Abelian gauge field theory of Tong and Wong where vortices generated from an extra gauge sector are used to realize magnetic impurities. It is seen that, in view of the fully coupled Einstein and gauge-matter equations, the presence of such cosmic strings in the form of topological defects is essential for gravitation. Asymptotic behavior of the string solutions can be precisely described to allow the derivation of a necessary and sufficient condition for the gravitational metric to be geodesically complete and an explicit calculation of the deficit angle proportional to the string tension, both stated in terms of string numbers, energy levels of broken symmetries, and the universal gravitational constant.

Topological defects in the form of domain walls, vortices, monopoles, and instantons play an important role in particle physics and condensed matter systems, and arise as a consequence of spontaneous symmetry breaking in various quantum field theory models. In cosmology, vortices give rise to locally concentrated structures commonly known as cosmic strings [1–3] which provide a seeding mechanism for galaxies to form in the early universe. That the occurrence of cosmic strings is topological comes from the nature of vacuum manifolds with broken symmetries, which leads to the onset of mixed states or impurities. Such mixed states give rise to energy concentrations around the spots where impurities take place, which, in view of the coupling of the Einstein equations, lead to curvature concentrations, and hence, non-uniformity of spacetime realizing a string structure. Although it is now known that cosmic strings arise in a broad range of theoretical contexts including superstring theory [4,5], it was first perceived in the Abelian Higgs model [1–3], which distinguishes itself as a prototypical minimal platform where cosmic strings

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appear as superconducting strings [6]. A salient feature of such cosmic strings is that the simplicity of the model enables a great amount of sharp insight [7] of the complicated gravitational behavior of cosmic strings to be obtained through a study of the solutions of the coupled Einstein and Abelian Higgs equations, thanks to the work in [8,9], unveiling a Bogomol'nyi–Prasad–Sommerfield (BPS) structure [10,11]. The purpose of this paper is to show that the methods of [8,9] may be extended to obtain exact results for cosmic strings generated from a recent product Abelian gauge field theory model of Tong and Wong [12], which uses heavy frozen vortices sitting in a different gauge group to realize magnetic impurities, so that the Einstein equations allow a complete resolution as well, thereby extending the zoo of models for which multiply distributed cosmic string solutions can be constructed and precisely understood.

Following [12], we consider two Abelian gauge fields, $\hat{A}_\mu$ and $\tilde{A}_\mu$, generated from the product gauge group $\hat{U}(1) \times \tilde{U}(1)$, and use q and p to denote two charged scalars carrying charge $(+1, -1)$ and $(0, +1)$, respectively, so that the gauge-covariant derivatives are given by

$$D_\mu q = \partial_\mu q - \mathrm{i}\hat{A}_\mu q + \mathrm{i}\tilde{A}_\mu q, \qquad D_\mu p = \partial_\mu p - \mathrm{i}\tilde{A}_\mu p. \tag{1}$$

Using the spacetime metric tensor $g_{\mu\nu}$ of signature $(+---)$ to raise and lower indices, we rewrite the Lagrangian density of the Tong–Wong model [12] as

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}g^{\mu\mu'}g^{\nu\nu'}\hat{F}_{\mu\nu}\hat{F}_{\mu'\nu'} - \frac{1}{4}g^{\mu\mu'}g^{\nu\nu'}\tilde{F}_{\mu\nu}\tilde{F}_{\mu'\nu'} + g^{\mu\nu}D_\mu q\overline{D_\nu q} + g^{\mu\nu}D_\mu p\overline{D_\nu p} \\ &- \frac{1}{2}\big(|q|^2 - \zeta\big)^2 - \frac{1}{2}\big(-|q|^2 + |p|^2 - \tilde{\zeta}\big)^2,\end{aligned} \tag{2}$$

where $\zeta, \tilde{\zeta} > 0$ are parameters, $\hat{F}_{\mu\nu}$, $\tilde{F}_{\mu\nu}$ are electromagnetic fields induced from gauge potentials $\hat{A}_\mu$, $\tilde{A}_\mu$. Varying the Einstein–Hilbert action

$$S = \int \left\{ \frac{1}{16\pi G}(R_g - 2\Lambda) + \mathcal{L} \right\} \sqrt{-g}\, \mathrm{d}x, \tag{3}$$

where R_g is the Ricci scalar curvature of the metric $g_{\mu\nu}$, $G > 0$ the universal gravitational constant, and Λ the cosmological constant, the equations of the motion of the action (3) are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}, \tag{4}$$

$$\frac{1}{\sqrt{-g}} D_\mu\big(g^{\mu\nu}\sqrt{-g} D_\nu q\big) = \big(|q|^2 - \zeta\big) q, \tag{5}$$

$$\frac{1}{\sqrt{-g}} D_\mu\big(g^{\mu\nu}\sqrt{-g} D_\nu p\big) = \big(-|q|^2 + |p|^2 - \tilde{\zeta}\big) p, \tag{6}$$

$$\frac{1}{\sqrt{-g}} \partial_{\mu'}\big(g^{\mu\nu}g^{\mu'\nu'}\sqrt{-g}\hat{F}_{\nu\nu'}\big) = \mathrm{i}g^{\mu\nu}(q\overline{D_\nu q} - \overline{q} D_\nu q), \tag{7}$$

$$\frac{1}{\sqrt{-g}} \partial_{\mu'}\big(g^{\mu\nu}g^{\mu'\nu'}\sqrt{-g}\tilde{F}_{\nu\nu'}\big) = \mathrm{i}g^{\mu\nu}(p\overline{D_\nu p} - \overline{p} D_\nu p) - \mathrm{i}g^{\mu\nu}(q\overline{D_\nu q} - \overline{q} D_\nu q), \tag{8}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ the energy–momentum tensor of the matter and gauge field sector given by

$$\begin{aligned}T_{\mu\nu} = &-g^{\mu'\nu'}\hat{F}_{\mu\mu'}\hat{F}_{\nu\nu'} - g^{\mu'\nu'}\tilde{F}_{\mu\mu'}\tilde{F}_{\nu\nu'} + D_\mu q\overline{D_\nu q} + \overline{D_\mu q}D_\nu q \\ &+ D_\mu p\overline{D_\nu p} + \overline{D_\mu p}D_\nu p - g_{\mu\nu}\mathcal{L}.\end{aligned} \tag{9}$$

As in [8,9] we look for straight time independent cosmic string solutions so that the spacetime is uniform along the time axis $x^0 = t$ and the x^3-direction and the line element takes the form $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - (\mathrm{d}x^3)^2 - g_{jk}\mathrm{d}x^j \mathrm{d}x^k$ $(j,k = 1,2)$ where now g_{jk} is the Riemannian metric tensor of an orientable 2-surface $\mathcal{M}$ with local coordinates x^1, x^2. Within such a metric the only nontrivial components of the Einstein tensor are $G_{00} = -G_{33} = -K_g$, where K_g is the Gauss curvature of the surface $(\mathcal{M}, \{g_{jk}\})$, which imposes constraints to the form of the energy–momentum tensor via (4). On the other hand, it is natural and compatible to assume that the gauge and scalar fields depend on the coordinates on $\mathcal{M}$ only and that the 0- and 3-components of the gauge fields are zero. Thus, we have $T_{03} = T_{0j} = T_{3j} = 0$ and $T_{00} = -T_{33} = \mathcal{H}$ immediately, where $\mathcal{H}$ is the Hamiltonian of the Tong–Wong model (2) given by

$$\mathcal{H} = -\mathcal{L} = \frac{1}{4}g^{jj'}g^{kk'}\hat{F}_{jk}\hat{F}_{j'k'} + \frac{1}{4}g^{jj'}g^{kk'}\tilde{F}_{jk}\tilde{F}_{j'k'} + g^{jk}D_j q\overline{D_k q} + g^{jk}D_j p\overline{D_k p} + \frac{1}{2}\big(|q|^2 - \zeta\big)^2 + \frac{1}{2}\big(-|q|^2 + |p|^2 - \tilde{\zeta}\big)^2. \tag{10}$$

To facilitate our computation, recall that the associated current densities

$$J_k^{(q)} = \frac{\mathrm{i}}{2}(q\overline{D_k q} - \overline{q}D_k q), \qquad J_k^{(p)} = \frac{\mathrm{i}}{2}(p\overline{D_k p} - \overline{p}D_k p), \tag{11}$$

satisfy the identities

$$\partial_j J_k^{(q)} - \partial_k J_j^{(q)} = \mathrm{i}(D_j q\overline{D_k q} - \overline{D_j q}D_k q) - \hat{F}_{jk}|q|^2 + \tilde{F}_{jk}|q|^2, \tag{12}$$

$$\partial_j J_k^{(p)} - \partial_k J_j^{(p)} = \mathrm{i}(D_j p\overline{D_k p} - \overline{D_j p}D_k p) - \tilde{F}_{jk}|p|^2. \tag{13}$$

Applying (12) and (13), we may rewrite (10) as

$$\begin{aligned}\mathcal{H} = {} & \frac{1}{4}g^{jj'}g^{kk'}\big(\hat{F}_{jk} \pm \epsilon_{jk}\big(|q|^2 - \zeta\big)\big)\big(\hat{F}_{j'k'} \pm \epsilon_{j'k'}\big(|q|^2 - \zeta\big)\big) \\ & + \frac{1}{4}g^{jj'}g^{kk'}\big(\tilde{F}_{jk} \pm \epsilon_{jk}\big(-|q|^2 + |p|^2 - \tilde{\zeta}\big)\big)\big(\tilde{F}_{j'k'} \pm \epsilon_{j'k'}\big(-|q|^2 + |p|^2 - \tilde{\zeta}\big)\big) \\ & + g^{jk}\big(D_j q \pm \mathrm{i}\epsilon_j^{j'}D_{j'}q\big)\overline{\big(D_k q \pm \mathrm{i}\epsilon_k^{k'}D_{k'}q\big)} \\ & + g^{jk}\big(D_j p \pm \mathrm{i}\epsilon_j^{j'}D_{j'}p\big)\overline{\big(D_k p \pm \mathrm{i}\epsilon_k^{k'}D_{k'}p\big)} \\ & \pm \frac{1}{2}\zeta\epsilon^{jk}\hat{F}_{jk} \pm \frac{1}{2}\tilde{\zeta}\epsilon^{jk}\tilde{F}_{jk} \pm \nabla_j\big(\epsilon^{jk}J_k^{(q)}\big) \pm \nabla_j\big(\epsilon^{jk}J_k^{(p)}\big),\end{aligned} \tag{14}$$

where ∇_j is the covariant derivative with respect to the metric g_{jk} over $\mathcal{M}$ and ϵ_{jk} is the Kronecker skew-symmetric tensor with $\epsilon_{12} = \sqrt{|g|}$ in which $|g| = \det(g_{jk})$.

In view of [12], we designate the $\hat{U}(1)$ and $\tilde{U}(1)$ magnetic fluxes by

$$\frac{1}{4\pi}\int_{\mathcal{M}} \epsilon^{jk}\hat{F}_{jk}\sqrt{|g|}\,\mathrm{d}x = \hat{k}, \qquad \frac{1}{4\pi}\int_{\mathcal{M}} \epsilon^{jk}\tilde{F}_{jk}\sqrt{|g|}\,\mathrm{d}x = \tilde{k}, \tag{15}$$

where $\hat{k}, \tilde{k}$ are some integers. Then we obtain from integrating (14) the energy lower bound

$$E = \int_{\mathcal{M}} \mathcal{H}\sqrt{|g|}\,\mathrm{d}x \geq 2\pi\big(\zeta|\hat{k}| + \tilde{\zeta}|\tilde{k}|\big). \tag{16}$$

It is clear that the lower bound in (16) is attained when the equations

$$\left.\begin{aligned}\hat{F}_{jk} \pm \epsilon_{jk}(|q|^2 - \zeta) &= 0,\\ \tilde{F}_{jk} \pm \epsilon_{jk}(-|q|^2 + |p|^2 - \tilde{\zeta}) &= 0,\\ D_j q \pm \mathrm{i}\epsilon_j^k D_k q &= 0,\\ D_j p \pm \mathrm{i}\epsilon_j^k D_k p &= 0,\end{aligned}\right\} \tag{17}$$

are satisfied for $j, k = 1, 2$ with $\hat{k} = \pm|\hat{k}|, \tilde{k} = \pm|\tilde{k}|$. These are the curved-space version of the BPS equations, originally derived in [12] when the spacetime is flat. As a consequence of (17), it is direct (although a bit cumbersome) to check that $T_{jk} = 0$ $(j, k = 1, 2)$, as would have naturally been anticipated [13]. Inserting this result into (4) we arrive at the vanishing cosmological constant condition, $\Lambda = 0$, which will be observed throughout the rest of this paper. Therefore, with such a reduction, the coupled Einstein–Tong–Wong equations stated as (4)–(8) become

$$K_g = 8\pi G\mathcal{H}, \tag{18}$$

$$\frac{1}{\sqrt{|g|}} D_j\big(g^{jk}\sqrt{|g|}D_k q\big) = \big(|q|^2 - \zeta\big)q, \tag{19}$$

$$\frac{1}{\sqrt{|g|}} D_j\big(g^{jk}\sqrt{|g|}D_k p\big) = \big(-|q|^2 + |p|^2 - \tilde{\zeta}\big)p, \tag{20}$$

$$\frac{1}{\sqrt{|g|}} \partial_{j'}\big(g^{jk}g^{j'k'}\sqrt{|g|}\hat{F}_{kk'}\big) = \mathrm{i}g^{jk}(q\overline{D_k q} - \bar{q}D_k q), \tag{21}$$

$$\frac{1}{\sqrt{|g|}} \partial_{j'}\big(g^{jk}g^{j'k'}\sqrt{|g|}\tilde{F}_{kk'}\big) = \mathrm{i}g^{jk}(p\overline{D_k p} - \bar{p}D_k p) - \mathrm{i}g^{jk}(q\overline{D_k q} - \bar{q}D_k q). \tag{22}$$

Defects or impurities occur at the spots where q or p vanishes. We now show that their presence is essential for gravitation as well. In fact, suppose q and p never vanish. Then, modulo a suitable gauge transformation, we may assume q and p are positive-valued functions. Thus, the real and imaginary parts in (19)–(22) decouple into

$$\Delta_g q = g^{jk}(\hat{A}_j - \tilde{A}_j)(\hat{A}_k - \tilde{A}_k)q + \big(q^2 - \zeta\big)q, \tag{23}$$

$$\frac{1}{\sqrt{|g|}} \partial_j\big(g^{jk}\sqrt{|g|}(\hat{A}_k - \tilde{A}_k)\big) = -2g^{jk}(\hat{A}_j - \tilde{A}_j)\partial_k \ln q, \tag{24}$$

$$\Delta_g p = g^{jk}\tilde{A}_j\tilde{A}_k q + \big(-q^2 + p^2 - \tilde{\zeta}\big)p, \tag{25}$$

$$\frac{1}{\sqrt{|g|}} \partial_j\big(g^{jk}\sqrt{|g|}\tilde{A}_k\big) = -2g^{jk}\tilde{A}_j\partial_k \ln p, \tag{26}$$

$$\frac{1}{\sqrt{|g|}} \partial_{j'}\big(g^{jk}g^{j'k'}\sqrt{|g|}\hat{F}_{kk'}\big) = -2g^{jk}(\hat{A}_k - \tilde{A}_k)q^2, \tag{27}$$

$$\frac{1}{\sqrt{|g|}} \partial_{j'}\big(g^{jk}g^{j'k'}\sqrt{|g|}\tilde{F}_{kk'}\big) = -2g^{jk}\tilde{A}_k p^2 + 2g^{jk}(\hat{A}_k - \tilde{A}_k)q^2. \tag{28}$$

Multiplying (27) by $\hat{A}_j$, (28) by $\tilde{A}_j$, and adding, we obtain

$$\begin{aligned}&\frac{1}{\sqrt{|g|}} \partial_{j'}\big(g^{j'k'}\sqrt{|g|}\big(g^{jk}\hat{A}_j\hat{F}_{kk'}\big)\big) + \frac{1}{2}g^{j'k'}g^{jk}\hat{F}_{jj'}\hat{F}_{kk'}\\ &\quad + \frac{1}{\sqrt{|g|}} \partial_{j'}\big(g^{j'k'}\sqrt{|g|}\big(g^{jk}\tilde{A}_j\tilde{F}_{kk'}\big)\big) + \frac{1}{2}g^{j'k'}g^{jk}\tilde{F}_{jj'}\tilde{F}_{kk'}\\ &= -2g^{jk}(\hat{A}_j - \tilde{A}_j)(\hat{A}_k - \tilde{A}_k)q^2 - 2g^{jk}\tilde{A}_j\tilde{A}_k p^2.\end{aligned} \tag{29}$$

Integrating (29) over $\mathcal{M}$, we have

$$\int_{\mathcal{M}} \left\{ \frac{1}{4} g^{j'k'} g^{jk} \hat{F}_{jj'} \hat{F}_{kk'} + \frac{1}{4} g^{j'k'} g^{jk} \tilde{F}_{jj'} \tilde{F}_{kk'} + g^{jk}(\hat{A}_j - \tilde{A}_j)(\hat{A}_k - \tilde{A}_k) q^2 + g^{jk} \tilde{A}_j \tilde{A}_k p^2 \right\} \sqrt{|g|} \mathrm{d}x = 0, \tag{30}$$

which leads to $\hat{A}_j = \tilde{A}_j \equiv 0$ for $j = 1, 2$. Consequently (23) becomes

$$\Delta_g q = q(q + \sqrt{\zeta})(q - \sqrt{\zeta}). \tag{31}$$

In view of $q > 0$ and the maximum principle we conclude that $q \equiv \sqrt{\zeta}$. Inserting this into (25) we find $p \equiv \sqrt{\zeta + \tilde{\zeta}}$ in a similar way. These results say the gauge-matter sector is trivial and $\mathcal{H} = 0$. Thus (18) implies $K_g = 0$. That is, $\mathcal{M}$ is flat and gravity is absent as well. This observation confirms the importance of mixed states to the occurrence of gravitation in the context of cosmic strings arising in the product Abelian gauge field theory of Tong and Wong [12], as that in the classical Abelian Higgs theory [7].

It is difficult to construct mixed-state solutions of the full system of the coupled equations (18)–(22). Fortunately, a big advantage we have here is that (17) implies (19)–(22). In other words, in order to solve the cosmic string equations (18)–(22) of the product Abelian gauge field theory model [12], it suffices to solve the BPS equations (17) coupled with the energy–curvature equation (18), which will be our next goal.

In view of (14) and (17), we have

$$\begin{aligned} \mathcal{H} &= \pm \frac{1}{2} \zeta \epsilon^{jk} \hat{F}_{jk} \pm \frac{1}{2} \tilde{\zeta} \epsilon^{jk} \tilde{F}_{jk} \pm \nabla_j \left(\epsilon^{jk} J_k^{(q)}\right) \pm \nabla_j \left(\epsilon^{jk} J_k^{(p)}\right) \\ &= -\zeta\left(|q|^2 - \zeta\right) - \tilde{\zeta}\left(-|q|^2 + |p|^2 - \tilde{\zeta}\right) + \frac{1}{2} \Delta_g |q|^2 + \frac{1}{2} \Delta_g |p|^2, \end{aligned} \tag{32}$$

where Δ_g is the Laplace–Beltrami operator induced from covariant derivative ∇_j such that $\Delta_g f = \nabla^j \nabla_j f = \frac{1}{\sqrt{|g|}} \partial_j (g^{jk} \sqrt{|g|} \partial_k f)$.

By virtue of [12], we use n and $\tilde{n}$ to denote the winding numbers of q and p, respectively. Without loss of generality we assume $n \geq 0$ and $\tilde{n} \geq 0$. Then $n, \tilde{n}$ are the algebraic numbers of zeros of q, p, respectively, which are related to the magnetic flux numbers $\hat{k}, \tilde{k}$ by $n = \hat{k} - \tilde{k}$, $\tilde{n} = \tilde{k}$ as indicated in [12]. The sets of zeros of q, p are denoted as

$$Z^{(q)} = \{z_{q,1}, \ldots, z_{q,n}\}, \qquad Z^{(p)} = \{z_{p,1}, \ldots, z_{p,\tilde{n}}\}. \tag{33}$$

These points are the centers of impurities or defects giving rise to locally concentrated gravitating vortices referred to as cosmic strings.

Set $u = \ln |q|^2$ and $v = \ln |p|^2$. It is standard that (17) may be recast into

$$\left.\begin{aligned} \Delta_g u &= 4\mathrm{e}^u - 2\mathrm{e}^v - 2(\zeta - \tilde{\zeta}) + 4\pi \textstyle\sum_{s=1}^{n} \delta_{z_{q,s}}, \\ \Delta_g v &= -2\mathrm{e}^u + 2\mathrm{e}^v - 2\tilde{\zeta} + 4\pi \textstyle\sum_{s=1}^{\tilde{n}} \delta_{z_{p,s}}, \end{aligned}\right\} \tag{34}$$

where δ_z denotes the Dirac function defined over the 2-surface $(\mathcal{M}, \{g_{jk}\})$ and concentrated at the point $z \in \mathcal{M}$.

To proceed further, we assume that the unknown metric g_{jk} is globally conformal to a known one, $g_{0,jk}$, so that $g_{jk} = \mathrm{e}^{\eta} g_{0,jk}$ $(j, k = 1, 2)$. Then we have the relations

$$-\Delta_{g_0} \eta + 2K_{g_0} = 2K_g \mathrm{e}^{\eta}, \qquad \Delta_g = \mathrm{e}^{-\eta} \Delta_{g_0}. \tag{35}$$

In view of the second relation in (35), we see that the system (34) becomes

$$\left.\begin{array}{l}\Delta_{g_0} u = \mathrm{e}^{\eta}(4\mathrm{e}^u - 2\mathrm{e}^v - 2(\zeta - \tilde{\zeta})) + 4\pi\sum_{s=1}^{n}\delta_{z_{q,s}},\\ \Delta_{g_0} v = \mathrm{e}^{\eta}(-2\mathrm{e}^u + 2\mathrm{e}^v - 2\tilde{\zeta}) + 4\pi\sum_{s=1}^{\tilde{n}}\delta_{z_{p,s}},\end{array}\right\} \tag{36}$$

where now the Dirac δ-functions are defined over the 2-surface $(\mathcal{M}, g_{0,jk})$ instead.

We first consider the case when $\mathcal{M}$ is compact. Use $|\mathcal{M}|_0$ to denote the volume of $\mathcal{M}$ with respect to the metric $g_{0,jk}$. Let u_0, v_0 be suitably superimposed functions of the solution [14] to

$$\Delta_{g_0} f = -\frac{4\pi}{|\mathcal{M}|_0} + 4\pi\delta_z, \quad z \in \mathcal{M}, \tag{37}$$

with z running over the zero sets $Z^{(q)}, Z^{(p)}$, stated in (33). Set $u = u_0 + U$ and $v = v_0 + V$ in (36). Then we get

$$\left.\begin{array}{l}\Delta_{g_0} U = \mathrm{e}^{\eta}(4\mathrm{e}^{u_0+U} - 2\mathrm{e}^{v_0+V} - 2(\zeta - \tilde{\zeta})) + \frac{4\pi n}{|\mathcal{M}|_0},\\ \Delta_{g_0} V = \mathrm{e}^{\eta}(-2\mathrm{e}^{u_0+U} + 2\mathrm{e}^{v_0+V} - 2\tilde{\zeta}) + \frac{4\pi \tilde{n}}{|\mathcal{M}|_0}.\end{array}\right\}. \tag{38}$$

Besides, from (18), (32), and (35), we have

$$\begin{aligned}\frac{1}{8\pi G}K_g &= \frac{\mathrm{e}^{-\eta}}{8\pi G}\left(-\frac{1}{2}\Delta_{g_0}\eta + K_{g_0}\right) = \mathcal{H}\\ &= -\zeta\left(\mathrm{e}^u - \zeta\right) - \tilde{\zeta}\left(-\mathrm{e}^u + \mathrm{e}^v - \tilde{\zeta}\right) + \frac{1}{2}\mathrm{e}^{-\eta}\Delta_{g_0}\mathrm{e}^u + \frac{1}{2}\mathrm{e}^{-\eta}\Delta_{g_0}\mathrm{e}^v.\end{aligned} \tag{39}$$

Combining (38) and (39), we arrive at

$$\Delta_{g_0}\left(-\frac{1}{8\pi G}\eta + \zeta U + (\zeta + \tilde{\zeta})V - \mathrm{e}^u - \mathrm{e}^v\right) = -\frac{K_{g_0}}{4\pi G} + \frac{4\pi}{|\mathcal{M}|_0}\left(\zeta n + (\zeta + \tilde{\zeta})\tilde{n}\right). \tag{40}$$

We claim that (40) is integrable.

In fact, integrating (38), we find

$$\int_{\mathcal{M}}\left(-|q|^2 + |p|^2 - \tilde{\zeta}\right)\sqrt{|g|}\,\mathrm{d}x = \int_{\mathcal{M}}\mathrm{e}^{\eta}\left(-\mathrm{e}^{u_0+U} + \mathrm{e}^{v_0+V} - \tilde{\zeta}\right)\sqrt{|g_0|}\,\mathrm{d}x = -2\pi\tilde{n}, \tag{41}$$

$$\int_{\mathcal{M}}\left(|q|^2 - \zeta\right)\sqrt{|g|}\,\mathrm{d}x = \int_{\mathcal{M}}\mathrm{e}^{\eta}\left(\mathrm{e}^{u_0+U} - \zeta\right)\sqrt{|g_0|}\,\mathrm{d}x = -2\pi(n + \tilde{n}), \tag{42}$$

which in view of (15) and (17) lead to $\hat{k} = n + \tilde{n}$, $\tilde{k} = \tilde{n}$, as already stated.

Thus, integrating (18), applying the saturated lower bound (16), and using the Gauss–Bonnet theorem, we have

$$2\pi\chi(\mathcal{M}) = \int_{\mathcal{M}} K_g\sqrt{|g|}\,\mathrm{d}x = 16\pi^2 G\left(\zeta n + (\zeta + \tilde{\zeta})\tilde{n}\right), \tag{43}$$

where $\chi(\mathcal{M})$ is the Euler characteristic of $\mathcal{M}$ which may be expressed as $\chi(\mathcal{M}) = 2(1 - m)$, in which $m = 0, 1, 2, \ldots$ is the genus of $\mathcal{M}$. Since the right-hand side of (43) is positive in nontrivial situations, we must have $m = 0$. In other words, $\mathcal{M}$ is topologically a 2-sphere, which will be our assumption from now on. Inserting $\chi(\mathcal{M}) = 2$ in (43) we see that the gravitational constant G and coupling parameters $\zeta, \tilde{\zeta}$ necessarily obey the constraint

$$\zeta n + (\zeta + \tilde{\zeta})\tilde{n} = \frac{1}{4\pi G}. \tag{44}$$

Use f to denote the right-hand side of (40). Integrating f over $(\mathcal{M}, \{g_{0,jk}\})$, applying the Gauss–Bonnet theorem, i.e., $\int_{\mathcal{M}} K_{g_0}\sqrt{|g_0|}\mathrm{d}x = 4\pi$, and using (44), we get

$$\int_{\mathcal{M}} f\sqrt{|g_0|}\mathrm{d}x = 0, \tag{45}$$

which proves that (40) is indeed consistent. Thus, there is a smooth function, say η_0, such that

$$-\frac{1}{8\pi G}\eta + \zeta U + (\zeta + \tilde{\zeta})V - \mathrm{e}^u - \mathrm{e}^v = -\frac{1}{8\pi G}\eta_0 + a, \quad \forall a \in \mathbb{R}. \tag{46}$$

For convenience, we may rewrite the relation (46) as

$$\eta = c + \eta_0 + 8\pi G\big(\zeta U + (\zeta + \tilde{\zeta})V - \mathrm{e}^{u_0+U} - \mathrm{e}^{v_0+V}\big) \equiv c + \sigma(x, U, V). \tag{47}$$

In summary, we see that the Einstein equations can be totally resolved and the cosmic strings are solutions of Eqs. (38) with the unknown conformal exponent η being determined by (47). In other words, we arrive at the closed system

$$\left.\begin{array}{l}\Delta_{g_0} U = \lambda \mathrm{e}^{\sigma(x,U,V)}(4\mathrm{e}^{u_0+U} - 2\mathrm{e}^{v_0+V} - 2(\zeta - \tilde{\zeta})) + \frac{4\pi n}{|\mathcal{M}|_0}, \\ \Delta_{g_0} V = \lambda \mathrm{e}^{\sigma(x,U,V)}(-2\mathrm{e}^{u_0+U} + 2\mathrm{e}^{v_0+V} - 2\tilde{\zeta}) + \frac{4\pi \tilde{n}}{|\mathcal{M}|_0},\end{array}\right\} \tag{48}$$

over a compact 2-surface $\mathcal{M}$ which is actually topologically a 2-sphere, where $\lambda > 0$ may be taken to be a free parameter.

Next we consider the situation when $\mathcal{M}$ is non-compact. For simplicity, we assume that $\mathcal{M}$ is conformally $\mathbb{R}^2$. That is, $S = \mathbb{R}^2$ and $g_{jk} = \mathrm{e}^{\eta}\delta_{jk}$. Now $g_{0,jk} = \delta_{jk}$ and $\Delta_{g_0} = \Delta$ is the usual Laplace operator on $\mathbb{R}^2$. Hence (36) becomes

$$\left.\begin{array}{l}\Delta u = \mathrm{e}^{\eta}(4\mathrm{e}^u - 2\mathrm{e}^v - 2(\zeta - \tilde{\zeta})) + 4\pi\sum_{s=1}^{n}\delta_{z_{q,s}}, \\ \Delta v = \mathrm{e}^{\eta}(-2\mathrm{e}^u + 2\mathrm{e}^v - 2\tilde{\zeta}) + 4\pi\sum_{s=1}^{\tilde{n}}\delta_{z_{p,s}},\end{array}\right\} \tag{49}$$

where the Dirac δ-functions are defined over $\mathbb{R}^2$. On the other hand, in (39), we have $K_{g_0} = 0$. Thus we obtain

$$\frac{1}{2}\mathrm{e}^{-\eta}\Delta\left(\frac{1}{8\pi G}\eta + \mathrm{e}^u + \mathrm{e}^v\right) = \zeta\left(\mathrm{e}^u - \zeta\right) + \tilde{\zeta}\left(-\mathrm{e}^u + \mathrm{e}^v - \tilde{\zeta}\right). \tag{50}$$

In order to have a more convenient description of the asymptotic behavior of the fields at infinity, we introduce the shifts

$$u \mapsto u + \ln\zeta, \qquad v \mapsto v + \ln(\zeta + \tilde{\zeta}). \tag{51}$$

Therefore (49) takes the updated form

$$\left.\begin{array}{l}\Delta u = \mathrm{e}^{\eta}(4\zeta\mathrm{e}^u - 2(\zeta + \tilde{\zeta})\mathrm{e}^v - 2(\zeta - \tilde{\zeta})) + 4\pi\sum_{s=1}^{n}\delta_{z_{q,s}}, \\ \Delta v = \mathrm{e}^{\eta}(-2\zeta\mathrm{e}^u + 2(\zeta + \tilde{\zeta})\mathrm{e}^v - 2\tilde{\zeta}) + 4\pi\sum_{s=1}^{\tilde{n}}\delta_{z_{p,s}},\end{array}\right\} \tag{52}$$

where u, v satisfy the normalized boundary condition $u = 0$, $v = 0$ at infinity of $\mathbb{R}^2$. Correspondingly, (50) is also modified into

$$\frac{1}{2}\mathrm{e}^{-\eta}\Delta\left(\frac{1}{8\pi G}\eta + \zeta\mathrm{e}^u + (\zeta + \tilde{\zeta})\mathrm{e}^v\right) = \zeta^2\left(\mathrm{e}^u - 1\right) + \tilde{\zeta}\left(-\zeta\mathrm{e}^u + (\zeta + \tilde{\zeta})\mathrm{e}^v - \tilde{\zeta}\right). \tag{53}$$

Combining (52) and (53), we obtain

$$\Delta\left(\frac{1}{8\pi G}\eta+\zeta\mathrm{e}^u+(\zeta+\tilde{\zeta})\mathrm{e}^v\right)$$
$$=\zeta\Delta u+(\zeta+\tilde{\zeta})\Delta v-4\pi\zeta\sum_{s=1}^{n}\delta_{z_{q,s}}-4\pi(\zeta+\tilde{\zeta})\sum_{s=1}^{\tilde{n}}\delta_{z_{p,s}}. \tag{54}$$

Thus

$$h\equiv\frac{1}{8\pi G}\eta-\zeta\big(u-\mathrm{e}^u\big)-(\zeta+\tilde{\zeta})\big(v-\mathrm{e}^v\big)$$
$$+\zeta\sum_{s=1}^{n}\ln|x-z_{q,s}|^2+(\zeta+\tilde{\zeta})\sum_{s=1}^{\tilde{n}}\ln|x-z_{p,s}|^2 \tag{55}$$

is an entire harmonic function over $\mathbb{R}^2$ which may well be assumed to be a constant. Therefore we see that the gravitational conformal factor is exactly determined to be given by the expression

$$\mathrm{e}^\eta=\lambda\left(\mathrm{e}^{\zeta(u-\mathrm{e}^u)+(\zeta+\tilde{\zeta})(v-\mathrm{e}^v)}\left(\prod_{s=1}^{n}|x-z_{q,s}|^2\right)^{-\zeta}\left(\prod_{s=1}^{\tilde{n}}|x-z_{p,s}|^2\right)^{-(\zeta+\tilde{\zeta})}\right)^{8\pi G}, \tag{56}$$

where $\lambda>0$ is an arbitrary constant.

Note that, since u, v satisfy (52), they behave like $\ln|x-z_{q,s}|^2$, $\ln|x-z_{p,s}|^2$ in a neighborhood of the points $z_{q,s}, z_{p,s}$, respectively. Consequently the conformal factor $\mathrm{e}^{\eta(x)}$ is a positive-valued smooth function over the full $\mathbb{R}^2$. In particular, the string points $z_{q,s}, z_{p,s}$ are not singular points of the gravitational metric generated from them.

In view of the boundary condition $u=0$, $v=0$ at infinity, we see that the conformal factor has the following precise asymptotic behavior

$$\mathrm{e}^{\eta(x)}=\mathrm{O}\big(|x|^{-16\pi G(\zeta n+[\zeta+\tilde{\zeta}]\tilde{n})}\big),\quad |x|\to\infty. \tag{57}$$

From (57), a number of interesting conclusions follow immediately. For example, we see that the solution gives rise to a geodesically complete metric if and only if the string numbers of the two species of strings, $n, \tilde{n}$, the energy levels of symmetry breaking scales, $\zeta, \tilde{\zeta}$, and the universal gravitational constant G satisfy the condition

$$\zeta n+(\zeta+\tilde{\zeta})\tilde{n}\le\frac{1}{8\pi G}. \tag{58}$$

Moreover, the apparent conical singularity at infinity gives rise to the deficit angle

$$\delta=16\pi^2G\big(\zeta n+[\zeta+\tilde{\zeta}]\tilde{n}\big)=8\pi GE, \tag{59}$$

where $E=2\pi(\zeta\hat{k}+\tilde{\zeta}\tilde{k})$ is the string energy or mass given as the lower bound in (16). Such a relation is in agreement that a cosmic string manifests its presence by displaying a deviation from the Euclidean geometry through the occurrence of a deficit angle whose magnitude depends on the string tension measured in the string energy or mass.

References

[1] T.W.B. Kibble, J. Phys. A, Math. Gen. 9 (1976) 1387;
T.W.B. Kibble, Phys. Rep. 69 (1980) 183.

[2] A. Vilenkin, E.P.S. Shellard, Cosmic Strings and Other Topological Defects, Cambridge Univ. Press, Cambridge, 1994.

[3] M. Hindmarsh, T.W.B. Kibble, Rep. Prog. Phys. 58 (1995) 477.
[4] S. Sarangi, S.-H. Tye, Phys. Lett. B 536 (2002) 185.
[5] E. Copeland, R.C. Myers, J. Polchinski, J. High Energy Phys. 6 (2004) 013.
[6] E. Witten, Nucl. Phys. B 249 (1985) 557.
[7] Y. Yang, Phys. Rev. Lett. 73 (1994) 10;
Y. Yang, Commun. Math. Phys. 170 (1995) 541;
Y. Yang, Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci. 453 (1997) 581.
[8] B. Linet, Gen. Relativ. Gravit. 20 (1988) 451.
[9] A. Comtet, G.W. Gibbons, Nucl. Phys. B 299 (1988) 719.
[10] E.B. Bogomol'nyi, Sov. J. Nucl. Phys. 24 (1976) 449.
[11] M.K. Prasad, C.M. Sommerfield, Phys. Rev. Lett. 35 (1975) 760.
[12] D. Tong, K. Wong, J. High Energy Phys. 1401 (2014) 090.
[13] Z. Hlousek, Nucl. Phys. B 397 (1993) 173.
[14] T. Aubin, Nonlinear Analysis on Manifolds: Monge–Ampére Equations, Springer, Berlin and New York, 1982.